# ABELIAN LENGTH CATEGORIES OF STRONGLY UNBOUNDED TYPE

#### HENNING KRAUSE

ABSTRACT. We discuss the notion of strongly unbounded type for abelian length categories; this is closely related to the Second Brauer–Thrall Conjecture for artin algebras. A new ingredient is the space of characters in the sense of Crawley-Boevey.

## 1. Introduction

The Second Brauer-Thrall Conjecture says that over infinite fields every finite dimensional algebra of infinite representation type is of strongly unbounded type [10]. In this note we take a fresh look at the conjecture. We consider abelian length categories which are Hom-finite over some commutative ring and discuss the analogue of 'strongly unbounded type' in this slightly more general setting. We use Crawley-Boevey's theory of characters [5, 6, 7] and further ideas from [9, 11].

A character for an abelian length category C is an integer valued function  $\mathsf{Ob}\,\mathsf{C}\to\mathbb{N}$  which is compatible with the exact structure (Definition 3.1). It turns out that the collection  $\mathsf{Sp}\,\mathsf{C}$  of irreducible characters carries a rich structure. It is a topological space via Ziegler's topology [20], and we investigate its basic properties. For instance, the Second Brauer–Thrall Conjecture amounts to the fact that  $\mathsf{Sp}\,\mathsf{C}$  is discrete if and only if C has only finitely many isomorphism classes of indecomposable objects (Theorem 3.4). A crucial property is the compactness of  $\mathsf{Sp}\,\mathsf{C}$ . There is an elegant proof of the Second Brauer–Thrall Conjecture when  $\mathsf{Sp}\,\mathsf{C}$  is compact. On the other hand, we point out an obstruction for the compactness of  $\mathsf{Sp}\,\mathsf{C}$  in terms of certain minimal closed subsets of  $\mathsf{Sp}\,\mathsf{C}$  (Proposition 5.1). This is illustrated by looking at modules over hereditary artin algebras (Corolloray 5.3). The closed subsets of  $\mathsf{Sp}\,\mathsf{C}$  we look at correspond to subcategories of C which are closed under subobjects; their relevance became apparent in recent work [14, 17]. Here, we show that each irreducible character is determined by such a closed subset (Corollary 4.4).

At this stage, not much seems to be known about the space Sp C. So we end this note with a list of open problems and hope to stimulate further progress.

# 2. Artin algebras

The following conjecture is the modification of the Second Brauer-Thrall Conjecture suggested by Crawley-Boevey in [4,  $\S1.6$ ]. Recall that the endolength of a module M is the length of M when viewed as a module over its endomorphism ring.

**Conjecture 2.1** (Brauer-Thrall II). Let A be an artin algebra of infinite representation type. Then for some  $n \in \mathbb{N}$  there are infinitely many non-isomorphic indecomposable A-modules which are of endolength n and of finite length over A.

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Without going into details, let us mention that this conjecture has been established for artin algebras over a field k when k is algebraically closed [1, 3], and more generally when k is perfect [2].

#### 3. Abelian length categories

Let k be a commutative ring and  $\mathsf{C}$  a k-linear abelian length category such that each morphism set in  $\mathsf{C}$  has finite length as a k-module. Suppose also that  $\mathsf{C}$  has only finitely many non-isomorphic simple objects.

**Definition 3.1** (Crawley-Boevey [7]). A *character* for C is a function  $\chi \colon \mathsf{Ob} \mathsf{C} \to \mathbb{N}$  satisfying the following:

- (1)  $\chi(X \oplus Y) = \chi(X) + \chi(Y)$  for all  $X, Y \in \mathsf{Ob}\,\mathsf{C}$ , and
- (2)  $\chi(X) + \chi(Z) \ge \chi(Y)$  for each exact sequence  $X \to Y \to Z \to 0$  in C.

A character  $\chi \neq 0$  is *irreducible* if  $\chi$  cannot be written as a sum of two non-zero characters. Note that any character can be written in a unique way as a finite sum of irreducible characters [6]. The *degree* of a character  $\chi$  is

$$\deg\chi=\sum_S\chi(S)$$

where S runs through a representative set of simple objects in C.

We denote by  $\operatorname{\mathsf{Sp}}\mathsf{C}$  the set of irreducible characters for  $\mathsf{C}$ . Fix a morphism  $\alpha\colon X\to Y$  in  $\mathsf{C}$  and  $n\in\mathbb{N}$ . For each character  $\chi$  set

$$\chi(\alpha) = \chi(X) - \chi(Y) + \chi(\operatorname{Coker} \alpha)$$

and let

$$\mathsf{U}_\alpha = \{\chi \in \mathsf{Sp}\,\mathsf{C} \mid \chi(\alpha) \neq 0\} \quad \text{and} \quad \mathsf{V}_{\alpha,n} = \{\chi \in \mathsf{Sp}\,\mathsf{C} \mid \chi(\alpha) \leq n\}.$$

The following lemma describes a topology on  $Sp\ C$ ; it is the analogue of Ziegler's topology on the isomorphism classes of indecomposable pure-injective modules over a ring [20].

**Lemma 3.2.** The subsets of the form  $U_{\alpha}$  where  $\alpha$  runs through all morphisms in C form a basis of open subsets for a topology on Sp C. The subsets of the form  $V_{\alpha,n}$  are closed. Moreover,

$$V_n = \{ \chi \in \mathsf{SpC} \mid \deg \chi < n \}$$

is closed and compact.

*Proof.* Consider the abelian category  $A = \mathsf{Fp}(\mathsf{C},\mathsf{Ab})$  of additive functors  $F \colon \mathsf{C} \to \mathsf{Ab}$  into the category  $\mathsf{Ab}$  of abelian groups that admit a *presentation* 

$$\operatorname{Hom}(Y, -) \longrightarrow \operatorname{Hom}(X, -) \longrightarrow F \longrightarrow 0.$$

A function  $\chi \colon \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$  is called *additive* provided that  $\chi(F) = \chi(F') + \chi(F'')$  if  $0 \to F' \to F \to F'' \to 0$  is an exact sequence. Restricting a function  $\chi \colon \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$  to  $\mathsf{Ob}\,\mathsf{C}$  by setting  $\chi(X) = \chi(\mathsf{Hom}(X,-))$  gives a bijection between the additive functions  $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$  and the characters  $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ . The inverse map takes a character  $\chi$  for  $\mathsf{C}$  to the function  $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$  which again we denote by  $\chi$  and which is defined by  $\chi(F) = \chi(\alpha)$  when F is presented by a morphism  $\alpha \colon X \to Y$  in  $\mathsf{C}$ ; this does not depend on the choice of  $\alpha$  by Schanuel's Lemma.

Now the assertions follow from properties of additive functions on A established in Lemmas B.5, B.6, and C.1 in [13]. For the compactness of  $V_n$  one uses that C has only finitely many simple objects.

Remark 3.3. To each irreducible character  $\chi \in \operatorname{Sp} \mathsf{C}$  corresponds a simple object  $S_{\chi}$  in some abelian quotient category of  $\mathsf{A}$ . The endomorphism ring of  $S_{\chi}$  is a division ring; it is an interesting invariant of  $\chi$  (see [13, Remark B.3]) but not relevant here.

The *character* and the *degree* of an object M in C are defined by

$$\chi_M(-) = \operatorname{length}_{\operatorname{End}(M)} \operatorname{Hom}(-, M)$$
 and  $\operatorname{deg} M = \operatorname{deg} \chi_M$ .

Note that  $\chi_M$  is irreducible when M is indecomposable, by [7, Theorem 3.6]. A character of the form  $\chi_M$  with M in C is called *finite*.

**Theorem 3.4.** Let C be a k-linear Hom-finite abelian category having only finitely many non-isomorphic simple objects. Consider the following statements.

- (1) The number of isomorphism classes of indecomposable objects in C is finite.
- (2) For every  $n \in \mathbb{N}$  the number of non-isomorphic indecomposable objects in C which are of degree n is finite.
- (3) Every irreducible character for C is finite.
- (4) The space of irreducible characters is discrete.

Then (2)–(4) are equivalent. Assuming in addition that Conjecture 2.1 holds, they are also equivalent to (1).

We need some preparations for the proof. Let  $A = Lex(C^{op}, Ab)$  be the category of left exact functors  $C^{op} \to Ab$ . This is a Grothendieck abelian category and C identifies via the Yoneda embedding taking X in C to Hom(-,X) with the full subcategory of finite-length objects in A; see [8, Chap. II] for details.

As before and following [7], we assign to each object M in A its character  $\chi_M$  (a function  $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N} \cup \{\infty\}$ ), and M is by definition endofinite if the values of  $\chi_M$  are finite.

**Theorem 3.5** (Crawley-Boevey). The assignment  $M \mapsto \chi_M$  induces a bijection between the isomorphism classes of indecomposable endofinite objects in A and the irreducible characters for C.

*Proof.* This is Theorem 3.6 in [7]; see also Proposition B.2 in [13].  $\Box$ 

The next lemma shows that the finite irreducible characters form a dense subset of  $\operatorname{\mathsf{Sp}} \mathsf{C}.$ 

**Lemma 3.6.** Let  $\chi_M$  be an irreducible character for C where M is an object in A. Write  $M = \operatorname{colim}_i M_i$  as a filtered colimit of objects in C and let U be the set of irreducible characters of the form  $\chi_N$  with N a direct summand of some  $M_i$ . Then  $\chi_M$  belongs to the closure of U.

*Proof.* Let  $\alpha: X \to Y$  be a morphism in  $\mathsf{C}$  and  $F: \mathsf{A} \to \mathsf{Ab}$  the corresponding functor with presentation

$$\operatorname{\mathsf{Hom}}(Y,-)\longrightarrow \operatorname{\mathsf{Hom}}(X,-)\longrightarrow F\longrightarrow 0.$$

Observe that  $\chi_M \in \mathsf{U}_\alpha$  iff  $F(M) \neq 0$ . We have  $\mathsf{colim}_i F(M_i) \xrightarrow{\sim} F(M)$ , and therefore  $\chi_M \in \mathsf{U}_\alpha$  implies  $\chi_N \in \mathsf{U}_\alpha$  for some  $\chi_N \in \mathsf{U}$ . Thus  $\chi_M$  belongs to the closure of  $\mathsf{U}$ .

**Corollary 3.7.** The finite irreducible characters form a dense subset of SpC.  $\square$ 

The following lemma identifies the isolated points in  $\operatorname{\mathsf{Sp}}\nolimits\mathsf{C}$ . Recall that a morphism  $\phi\colon M\to N$  in  $\mathsf{C}$  is left almost split if  $\phi$  is not a split monomorphism and every morphism  $M\to N'$  in  $\mathsf{C}$  which is not a split monomorphism factors through  $\phi$ .

**Lemma 3.8.** Let  $\chi \in \operatorname{Sp} C$ . Then  $\{\chi\}$  is open if and only if there is a left almost split morphism  $M \to N$  in C with  $\chi = \chi_M$ .

*Proof.* Suppose first that  $\{\chi\} = \mathsf{U}_{\alpha}$  for some morphism  $\alpha \colon X \to Y$  in  $\mathsf{C}$ . It follows from Corollary 3.7 that  $\chi = \chi_M$  for some indecomposable object M in  $\mathsf{C}$ . Choose a morphism  $\beta \colon X \to M$  which does not factor through  $\alpha$  and form the following pushout.

$$X \xrightarrow{\alpha} Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$M \longrightarrow N$$

It is easily checked that the morphism  $M \to N$  is left almost split. Conversely, if  $\alpha \colon M \to N$  is left almost split in C, then  $U_{\alpha} = \{\chi_M\}$ .

Proof of Theorem 3.4. There are two cases. Suppose first that there is a simple object S that does not embed into an injective object in C. Let E be an injective envelope of S in Lex( $C^{op}$ , Ab) and write  $E = \bigcup_i E_i$  as the directed union of nonzero objects in C. Note that E has degree one. Each  $E_i$  has a simple socle and is therefore indecomposable of degree one as well. On the other hand, the lengths of the  $E_i$  are unbounded. The object E yields an irreducible character  $\chi_E$  for C which lies in the closure of the  $\chi_{E_i}$ , by Lemma 3.6. Thus the space  $Sp\ C$  is indiscrete. It follows that each of (1)–(4) does not hold.

Now suppose that  $\mathsf{C}$  has enough injective objects. Let E be an injective cogenerator and A its endomorphism ring. Then A is an artin algebra and the functor  $\mathsf{Hom}(-,E)$  identifies  $\mathsf{C}$  with the category of finite-length A-modules. Note that each indecomposable object in  $\mathsf{C}$  is the source of a left almost split morphism.

- $(3) \Leftrightarrow (4)$ : The finite characters in  $\operatorname{\mathsf{SpC}}$  are precisely the isolated points, by Lemma 3.8. Thus  $\operatorname{\mathsf{SpC}}$  is discrete iff all irreducible characters are finite.
- $(3) \Rightarrow (2)$ : The set  $V_n$  of irreducible characters of degree at most n form a closed subset of  $\operatorname{Sp} C$  which is compact, by Lemma 3.2. Every character in  $V_n$  is finite and therefore isolated, by Lemma 3.8. Thus  $V_n$  is a finite set.
- $(2)\Rightarrow (3)$ : Suppose there is an irreducible character  $\chi$  which is not finite. Using the correspondence between irreducible characters for C and indecomposable endofinite A-modules [5, §5.3], it follows that  $\chi$  corresponds to an endofinite A-module which is not of finite length. This implies that there are infinitely many non-isomorphic indecomposable finite-length A-modules of some fixed endolength; see [5, §9.6]. For each A-module M we have

$$\deg M \leq \chi_M(A/\operatorname{rad} A) \leq \chi_M(A) = \operatorname{endol} M.$$

Thus for some  $n \in \mathbb{N}$  the number of non-isomorphic indecomposable objects in C which are of degree n is infinite.

- $(1) \Rightarrow (2)$ : Clear.
- (2)  $\Rightarrow$  (1): This follows from Conjecture 2.1, using that the degree of an A-module is bounded by its endolength.

Remark 3.9. The original notion of 'strongly unbounded type' asks for infinitely many  $n \in \mathbb{N}$  such that the number of non-isomorphic indecomposable objects of length n is infinite [10]. We cannot expect to have infinitely many such n when length is replaced by degree as in Theorem 3.4. To see this, consider the category of finitely generated torsion modules over any discrete valuation domain: all indecomposables are of degree one.

## 4. Subobject-closed subcategories

In this section we fix a k-linear Hom-finite abelian category  $\mathsf{C}$ . A full additive subcategory  $\mathsf{D}\subseteq \mathsf{C}$  is called  $\operatorname{subobject-closed}$  if each subobject of an object in  $\mathsf{D}$  belongs to  $\mathsf{D}$ . Note that in this case the inclusion  $\mathsf{D}\to\mathsf{C}$  has a left adjoint; it takes an object  $X\in\mathsf{C}$  to X/U where  $U\subseteq X$  is the minimal subobject with  $X/U\in\mathsf{D}$ . Thus  $\mathsf{D}$  is a category having cokernels and we can define characters  $\mathsf{Ob}\,\mathsf{D}\to\mathbb{N}$  as before.

**Lemma 4.1.** Let  $D \subseteq C$  be a subobject-closed subcategory and  $p: C \to D$  be the left adjoint of the inclusion. The map  $\operatorname{Sp} D \to \operatorname{Sp} C$  sending  $\chi$  to  $\chi \circ p$  identifies  $\operatorname{Sp} D$  with the closure of  $\{\chi_M \in \operatorname{Sp} C \mid M \in D\}$ .

*Proof.* Adapt the proof of Proposition 2.2 in [14].

A subset  $U \subseteq Sp\ C$  is said to be *subobject-closed* if  $U = Sp\ D$  for some subobject-closed additive subcategory  $D \subseteq C$ .

The subobject-closed subcategories of C form a set which is partially ordered by inclusion. An intersection  $\bigcap_{\alpha} C_{\alpha}$  of subobject-closed additive subcategories  $C_{\alpha} \subseteq C$  is again subobject-closed. Moreover, adapting the proof of [14, Corollary 2.3], we have

$$\operatorname{\mathsf{Sp}}(\bigcap_{\alpha}\mathsf{C}_{\alpha})=\bigcap_{\alpha}\operatorname{\mathsf{Sp}}\mathsf{C}_{\alpha}.$$

Thus we can define for each  $\chi \in \operatorname{Sp} \mathsf{C}$  the subobject-closed subset

$$\operatorname{sub}\chi=\bigcap_{\chi\in\mathsf{U}}\mathsf{U}$$

where U runs through all subobject-closed subsets  $U \subseteq Sp C$ .

**Theorem 4.2.** Let  $\chi, \psi$  be irreducible characters for C. Then

$$\operatorname{sub} \chi = \operatorname{sub} \psi \quad \iff \quad \chi = \psi.$$

The proof is based on the following lemma.

**Lemma 4.3.** Let M be an indecomposable endofinite object in  $A = Lex(C^{op}, Ab)$  and  $\phi: M \to \coprod_I M$  a monomorphism. Then at least one component of  $\phi$  is invertible

*Proof.* Consider in A the colimit of the chain of monomorphisms

$$M \stackrel{\phi}{\longrightarrow} \coprod_I M \xrightarrow{\coprod_I \phi} \coprod_{I^2} M \xrightarrow{\coprod_I \coprod_I \coprod_I \phi} \coprod_{I^3} M \xrightarrow{\coprod_I \coprod_I \coprod_I \bigoplus_I \phi} \cdots$$

and write this as morphism  $\bar{\phi} \colon M \to \coprod_J M$ . Denote by E the endomorphism ring of M; it is a local ring with  $\bigcap_{n \geq 0} \operatorname{rad} E = 0$ . This follows from [8, Prop. IV.14] since M can be identified with an indecomposable injective object of a locally finite abelian category; see [13, Proposition B.2]. If each component of  $\phi$  belongs to rad E, then each component of  $\bar{\phi}$  belongs  $\bigcap_{n \geq 0} \operatorname{rad} E$ . Thus there is one component of  $\phi$  which is invertible.

Proof of Theorem 4.2. As before, we work in the category  $\mathsf{A} = \mathsf{Lex}(\mathsf{C}^{\mathrm{op}},\mathsf{Ab})$ . For  $M \in \mathsf{A}$  let  $\mathsf{sub}\,M$  denote the subobject-closed subcategory of  $\mathsf{C}$  consisting of all finite-length subobjects of finite coproducts of copies of M. Note that M is a filtered colimit of objects in  $\mathsf{sub}\,\mathsf{M}$ .

Now fix indecomposable endofinite objects M and N in A. Then

$$\operatorname{\mathsf{sub}} \chi_M = \operatorname{\mathsf{Sp}}(\operatorname{\mathsf{sub}} M)$$

and therefore

$$\operatorname{sub} \chi_M \subseteq \operatorname{sub} \chi_N \iff \operatorname{sub} M \subseteq \operatorname{sub} N.$$

Suppose that  $\operatorname{sub}\chi_M\subseteq\operatorname{sub}\chi_N$ . Write  $M=\operatorname{colim}_i M_i$  as a filtered colimit of finite-length objects and choose for each i a monomorphism  $\alpha_i\colon M_i\to N_i$  into a finite coproduct of copies of N. Note that the coproducts of copies of N form a subcategory of A which is closed under filtered colimits and products, since N is endofinite; see [7, §3] or [12, Corollary 10.5]. Thus the  $\alpha_i$  induce a monomorphism

$$M = \operatornamewithlimits{colim}_i M_i \longrightarrow \operatornamewithlimits{colim}_i (\prod_{i \to j} N_j) = \coprod_I N$$

for some index set I.

Now suppose that  $\operatorname{\mathsf{sub}} \chi_M = \operatorname{\mathsf{sub}} \chi_N$ . Thus there is also a monomorphism  $N \to \coprod_J M$  for some index set J. Composing these morphisms yields a monomorphism

$$\phi \colon M \longrightarrow \coprod_{I} \coprod_{J} M.$$

Each component of  $\phi$  is an endomorphism of M that factors through a coproduct of copies of N, and it follows from Lemma 4.3 that at least one component is invertible. Thus  $M \cong N$  and therefore  $\chi_M = \chi_N$ .

Corollary 4.4. The set Sp C is partially ordered by

$$\chi \subseteq \psi \quad \iff \quad \operatorname{sub} \chi \subseteq \operatorname{sub} \psi.$$

In particular, each  $\chi \in \operatorname{Sp} C$  is uniquely determined by the set of finite characters in  $\operatorname{sub} \chi$ .

*Proof.* The first part is clear from Theorem 4.2. For the second part, observe that  $\operatorname{\mathsf{sub}} \chi$  is the closure of the subset of finite characters in  $\operatorname{\mathsf{sub}} \chi$  by Lemma 4.3.

Let us rephrase Theorem 4.2 for module categories, using the correspondence between characters and endofinite modules  $[5, \S 5.3]$ . For a module M, let  $\mathsf{sub}\,M$  denote the category consisting of all finite-length submodules of finite direct sums of copies of M.

**Corollary 4.5.** Two indecomposable endofinite modules M and N over an artin algebra are isomorphic if and only if  $\mathsf{sub}\,M = \mathsf{sub}\,N$ .

### 5. Compactness

We keep the setting of the previous section and discuss the compactness of  $\operatorname{\mathsf{Sp}}\mathsf{C}$ . More specifically, the fact that each irreducible character is determined by a subobject-closed subset of  $\operatorname{\mathsf{Sp}}\mathsf{C}$  raises the question when a subset of  $\operatorname{\mathsf{Sp}}\mathsf{C}$  is of the form  $\operatorname{\mathsf{sub}}\chi$  for some character  $\chi$ . We have the following criterion.

**Proposition 5.1.** Suppose that each indecomposable object in C is the source of a left almost split morphism. Let  $U \subseteq Sp\ C$  be an infinite subobject-closed subset which is minimal with respect to this property. Then the following conditions are equivalent.

- (1) U is compact.
- (2) U contains an infinite character.
- (3) U is of the form  $\sup \chi$  for some character  $\chi$ .

We need the following lemma.

**Lemma 5.2.** Let  $U \subseteq \operatorname{Sp} C$  and  $\chi \in \operatorname{Sp} C$ . Choose indecomposable endofinite objects  $(M_i)_{i \in I}$  and M in  $A = \operatorname{Lex}(C^{\operatorname{op}}, \operatorname{Ab})$  such that  $U = \{\chi_{M_i} \mid i \in I\}$  and  $\chi = \chi_M$ . Then the following conditions are equivalent.

- (1) The character  $\chi$  belongs to the closure of U.
- (2) The object M belongs to the smallest subcategory of A closed under products, filtered colimits, pure subobjects, and containing all  $M_i$ .

*Proof.* This follows from Corollary 4.6 and Theorem 6.2 in [12].  $\Box$ 

*Proof of Proposition 5.1.* As before, we work in the category  $A = Lex(C^{op}, Ab)$ . We denote by D the subobject-closed subcategory of C such that U = SpD.

- $(1) \Rightarrow (2)$ : Each finite character in SpC is isolated, by Lemma 3.8. It follows that U contains an infinite character when U is infinite and compact.
- $(2) \Rightarrow (3)$ : Let  $\chi \in U$  be infinite. Then  $\mathsf{sub}\,\chi \subseteq \mathsf{U}$ , and the minimality of  $\mathsf{U}$  implies equality.
- $(3)\Rightarrow (1)$ : Let  $\mathsf{U}=\mathsf{sub}\,\chi$  with  $\chi=\chi_M$  for some indecomposable endofinite object M in A. It suffices to show that each infinite closed subset  $\mathsf{V}\subseteq \mathsf{U}$  contains  $\chi$ . Thus we choose such an infinite subset  $\mathsf{V}$ . The minimality of  $\mathsf{U}$  implies that the smallest subobject-closed subset of  $\mathsf{Sp}\,\mathsf{C}$  containing  $\mathsf{V}$  coincides with  $\mathsf{U}$ . Write  $M=\mathsf{colim}_i\,M_i$  as a filtered colimit of objects in  $\mathsf{D}$ . It follows that for each i there is a monomorphisms  $M_i\to N_i$  with  $N_i$  a finite coproduct of indecomposable objects  $N\in\mathsf{C}$  such that  $\chi_N\in\mathsf{V}$ . There are also monomorphisms  $N_i\to P_i$  such that each  $P_i$  is a finite coproduct of copies of M, since  $N_i\in\mathsf{D}=\mathsf{sub}\,M$ . These monomorphisms induce a pair of monomorphisms

$$M = \operatornamewithlimits{colim}_i M_i \longrightarrow \operatornamewithlimits{colim}_i (\prod_{i \to j} N_j) \longrightarrow \operatornamewithlimits{colim}_i (\prod_{i \to j} P_j) = \coprod_I M$$

for some index set I, as in the proof of Theorem 4.2. It follows from Lemma 4.3 that at least one component of this composite is invertible. Thus  $\chi$  belongs to V, by Lemma 5.2. We conclude that U is compact.

There is an interesting consequence for the space of irreducible characters on the category of finite-length modules over a hereditary artin algebra. I am grateful to Claus Michael Ringel for providing the key observation.

**Corollary 5.3.** Let A be a connected hereditary artin algebra and mod A be the category of finite-length A-modules. Then the space  $\mathsf{Sp}(\mathsf{mod}\,A)$  is compact if and only if A is of finite or tame representation type.

*Proof.* The assertion is clear from Theorem 3.4 when A is of finite representation type, and it follows by inspection of the Ziegler spectrum when A is tame, see [15, 16]. Now suppose that A is of wild representation type and denote by C the category of preprojective modules; it is minimal among the subobject-closed subcategories of  $\operatorname{\mathsf{mod}} A$  which are of infinite type, by [17, Example 1]. There is no indecomposable endofinite A-module of infinite length which is a union of modules from C, by [18]. Thus the corresponding subset  $\operatorname{\mathsf{Sp}} C$  is not compact, by Proposition 5.1. It follows that  $\operatorname{\mathsf{Sp}}(\operatorname{\mathsf{mod}} A)$  is not compact.

# 6. Some open Problems

Not much seems to be known about the space  $\operatorname{\mathsf{Sp}}\mathsf{C}$  of characters for a length category  $\mathsf{C}$ . For instance, we need a better understanding of the interplay between finite and infinite characters. In this section we address some open problems. We fix a k-linear Hom-finite abelian category  $\mathsf{C}$  having only finitely many non-isomorphic simple objects.

The support of a character. It would be interesting to find for each irreducible character  $\chi$  some appropriate set of finite irreducible characters supporting  $\chi$ . More specifically, we ask the following.

**Question 6.1.** Is every irreducible character of degree n in the closure of a set of finite irreducible characters of degree at most n?

**Subobject-closed subsets.** The lattice of subobject-closed subsets of SpC contains a lot of information. For instance, the space SpC itself embeds naturally into this lattice, by Theorem 4.2. This motivates the following question.

**Question 6.2.** When is a subobject-closed subsets of SpC of the form  $\mathsf{sub}\,\chi$  for some irreducible character  $\chi$ ?

There is a compactness result for subobject-closed subcategories which is due to Ringel [17]; his proof uses properties of the Gabriel–Roiter measure. An alternative proof in [14] is based on the compactness of the Ziegler spectrum.

A full additive subcategory  $D \subseteq C$  is of *infinite type* if there are infinitely many isomorphism classes of indecomposable objects in D.

**Proposition 6.3.** Suppose that each indecomposable object in C is the source of a left almost split morphism. Then each subobject-closed additive subcategory of C that is of infinite type contains one which is minimal among all subobject-closed additive subcategories of infinite type.

*Proof.* Adapt the proof of Corollary 4.3 in [14].

From this proposition it follows that each infinite subobject-closed subset of  $\operatorname{\mathsf{Sp}}\nolimits\mathsf{C}$  contains a minimal one.

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**Question 6.4.** When does a minimal infinite subobject-closed subset of Sp C contain an infinite character?

Note that such a minimal infinite subobject-closed subset with an infinite character  $\chi$  is necessarily of the form  $\operatorname{\mathsf{sub}} \chi$ , by Proposition 5.1.

Remark 6.5. Let  $D \subseteq C$  be a minimal subobject-closed subcategory of infinite type. Then for each  $n \in \mathbb{N}$  the number of isomorphism classes of indecomposable objects of length n in D is finite; see [17, Theorem 2].

Compactness. When SpC is compact, one gets an elegant proof of the Second Brauer–Thrall Conjecture, because an infinite space cannot be discrete; see Theorem 3.4. This observation motivates the following question.

**Question 6.6.** When is Sp C compact?

Even when  $\operatorname{\mathsf{Sp}}\nolimits \mathsf{C}$  is not compact, we can ask for suitable subsets which are compact. Here is one possible question.

**Question 6.7.** When is a subset of the form  $\mathsf{sub}\,\chi$  compact?

Partial orders on the set of characters. The set of characters  $Ob C \to \mathbb{N}$  is partially ordered by

$$\chi \leq \psi \iff \chi(X) \leq \psi(X) \text{ for all } X \in \mathsf{Ob} \mathsf{C}.$$

Observe that for each character  $\psi$  the set

$$\mathsf{U}_{\psi} = \{ \chi \in \mathsf{SpC} \mid \chi \leq \psi \}$$

is closed and compact; this follows from Lemma 3.2.

From Theorem 4.2 it follows that the inclusion relation for  $\operatorname{\mathsf{sub}} \chi$  provides another partial order on the set of irreducible characters.

## **Question 6.8.** How are the partial orders $\leq$ and $\subseteq$ on Sp C related?

We refer to [14, 19] for more details on these partial orders.

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FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, D-33501 BIELEFELD, GERMANY E-mail address: hkrause@math.uni-bielefeld.de